Appendix to
"The Boxtree: Enabling Real-Time and Exact Collision Detection"

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Complete s- and t-Tables for the Boxtree-Algorithm

This appendix provides the complete tables needed for box-splitting while maintaining their intersection status, This is the basic step of the simultaneous traversal of two boxtrees thus implementing a divide-&-conquer algorithm which can be endowed with various semantics.

Terms and notations. We have to define a few terms and notations. In the following, a box A will be represented by a point p, three unit vectors b^x, b^y, b^z, and Δ_x^A, Δ_y^A, Δ_z^A (see Figure 1). The point p will be called the *origin of the box* A; it will be maintained in the coordinate system of A and in the coordinate system of B. The vectors $b^{\cdot}$ describe always the box axes in the other box' coordinate system.

Similarly, the box B is represented by b^x, b^y, b^z, Δ_x^B, Δ_y^B, Δ_z^B, and q.

The two planes which are perpendicular to the local x-axis b^x will be called *x-planes*; in particular, the x-plane which contains p will be called *xl-plane*; the x-plane which goes through $p + (\Delta_x^A, 0, 0)$ will be called *xh-plane*. Similarly, we define y- and z-planes.

Edges are grouped into three families. Each family consists of the four edges of the box which are parallel to each other. They will be called *x-*, *y-*, and *z-edges*.

Given two boxes A and B, we will check if they intersect each other by calculating a parameter interval of every edge of A corresponding to that part of the edge which is inside B, and vice versa. Line parameters of edges of A will be denoted by t's, those of B by s's. The line interval of a clipped edge will be called *t-interval*. The line interval of edge $x1$ of A will be denoted by $T^{x1} = [T_{\min}^{x1}, T_{\max}^{x1}]$. Line parameters are always with respect to p or q, resp., i.e., $t_{\alpha 0} = 0$ is the point p for $\alpha \in \{x, y, z\}$.

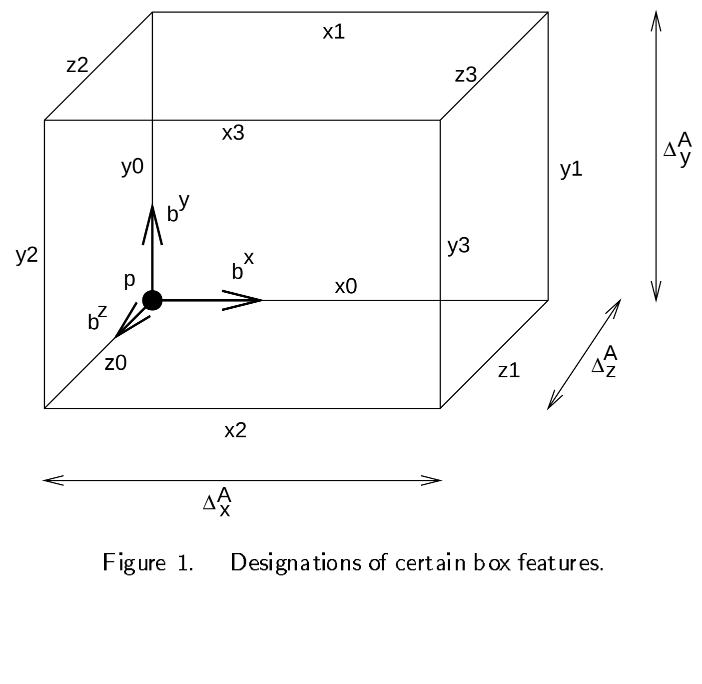

Figure 1. Designations of certain box features.

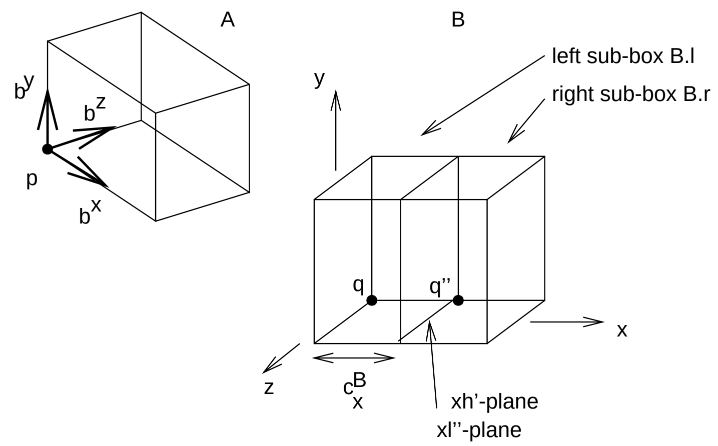

Figure 2. Splitting box B yields new line parameters for edges of A; cut-plane perpendicular to x-edges of B.

t^{x0}_{xl} denotes that line parameter where edge $x0$ of A intersects with the xl-plane of B; all other t- and s-parameters are defined analogously.

First intersection test. This paragraph describes the procedure of the first intersection test of the roots of two boxtrees.

First, we compute the line intervals of the x-edges of A clipped at B (see Figure 2). To do that, we need to calculate the line intervals $t^{xi}_{::}$ of all edges intersected with all planes of B. Simple calculus yields $t^{x0}_{xl} = \frac{q_x - p_x}{b^x_x}$. Similarly,

$$t^{x0}_{xh} = \frac{q_x - p_x + \Delta^B_x}{b^x_x} = t^{x0}_{xl} + \frac{\Delta^B_x}{b^x_x}$$

which exploits the fact that the faces xl and xh are parallel. We can also use the fact that all x-edges are parallel:

$$t^{x1}_{xl} = \frac{(q - (p + \Delta^A_y b^y)) \cdot (-1, 0, 0)}{b^x \cdot (-1, 0, 0)} = t^{x0}_{xl} - \Delta^A_y \frac{b^y_x}{b^x_x}$$

Analogously, we can calculate all the other line parameters, which can be summarized in the following table:

$t^{\cdot\cdot\rightarrow}_{\cdot\cdot\downarrow}$	$x0$	$x1$	$x2$	$x3$
xl	$\frac{q_x - p_x}{b^x_x}$	$t^{x0}_{xl} - \Delta^A_y \frac{b^y_x}{b^x_x}$	$t^{x0}_{xl} - \Delta^A_z \frac{b^z_x}{b^x_x}$	$t^{x1}_{xl} - \Delta^A_z \frac{b^z_x}{b^x_x}$
xh	$t^{x0}_{xl} - \Delta^A_y \frac{b^y_x}{b^x_x}$	$t^{x0}_{xh} - \Delta^A_y \frac{b^y_x}{b^x_x}$	$t^{x0}_{xh} - \Delta^A_z \frac{b^z_x}{b^x_x}$	$t^{x1}_{xh} - \Delta^A_z \frac{b^z_x}{b^x_x}$
yl	$\frac{q_y - p_y}{b^x_y}$	$t^{x0}_{yl} - \Delta^A_y \frac{b^y_y}{b^x_y}$	$t^{x0}_{yl} - \Delta^A_z \frac{b^z_y}{b^x_y}$	$t^{x1}_{yl} - \Delta^A_z \frac{b^z_y}{b^x_y}$
yh	$t^{x0}_{yl} - \Delta^A_y \frac{b^y_y}{b^x_y}$	$t^{x0}_{yh} - \Delta^A_y \frac{b^y_y}{b^x_y}$	$t^{x0}_{yh} - \Delta^A_z \frac{b^z_y}{b^x_y}$	$t^{x1}_{yh} - \Delta^A_z \frac{b^z_y}{b^x_y}$
zl	$\frac{q_z - p_z}{b^x_z}$	$t^{x0}_{zl} - \Delta^A_y \frac{b^y_z}{b^x_z}$	$t^{x0}_{zl} - \Delta^A_z \frac{b^z_z}{b^x_z}$	$t^{x1}_{zl} - \Delta^A_z \frac{b^z_z}{b^x_z}$
zh	$t^{x0}_{zl} - \Delta^A_y \frac{b^y_z}{b^x_z}$	$t^{x0}_{zh} - \Delta^A_y \frac{b^y_z}{b^x_z}$	$t^{x0}_{zh} - \Delta^A_z \frac{b^z_z}{b^x_z}$	$t^{x1}_{zh} - \Delta^A_z \frac{b^z_z}{b^x_z}$

We will call this table a *t-table* for the x-edges of A. We need three t-tables for the three sets of x-, y-, and z-edges of A. There will be similar tables for the edges of B, which we will call *s-tables*.

Descending the box-tree. For a given pair (a, b) of boxes, all the information on their intersection status is given by a set of $2 \times 3 \times 4$ line parameter intervals. The basic step of the traversal is the test "`a,b.l` *intersect*" and "`a,b.r` *intersect*". We will do this by bisecting the box b into its left and right sub-box, which is equivalent to computing two new sets of $2 \times 3 \times 4$ line parameter intervals, one

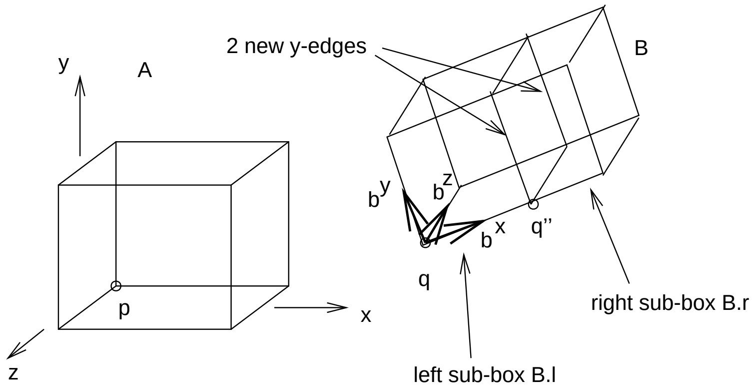

Figure 3. Splitting box B yields new line parameters for edges of B; cut-plane perpendicular to x-edges of B.

describing $(a, b.l)$, the other describing $(a, b.r)$. In this sub-section we will describe the splitting of the box b; splitting a is quite analogous.

New line parameters are always computed in the other box's local coordinate system. We will denote the new t-values (where a line intersects with a face) by $'t_{\cdot\cdot}^{\cdot\cdot}$ for edges of a clipped at $b.l$, and by $''t_{\cdot\cdot}^{\cdot\cdot}$ for edges of A clipped at $b.r$, resp.

Splitting B at c_x^B. All $'t$-values equal t-values except for $'t_{xh}^{\cdot\cdot}$ and $''t_{xl}^{\cdot\cdot}$ (see Figure 2). Fortunately, $'t_{xh}^{\cdot\cdot} = ''t_{xl}^{\cdot\cdot}$. So we have to compute the following values $'t_{xh}^{\alpha i}$:

$\alpha \backslash i$	0	1	2	3
x	$\frac{q''_x - p_x}{b_x^x}$	$'t_{xh}^{x0} - \Delta_y^A \frac{b_x^y}{b_x^x}$	$'t_{xh}^{x0} - \Delta_z^A \frac{b_x^z}{b_x^x}$	$'t_{xh}^{x1} - \Delta_z^A \frac{b_x^z}{b_x^x}$
y	$\frac{q''_x - p_x}{b_x^y}$	$'t_{xh}^{y0} - \Delta_x^A \frac{b_x^x}{b_x^y}$	$'t_{xh}^{y0} - \Delta_z^A \frac{b_x^z}{b_x^y}$	$'t_{xh}^{y1} - \Delta_z^A \frac{b_x^z}{b_x^y}$
z	$\frac{q''_x - p_x}{b_x^z}$	$'t_{xh}^{z0} - \Delta_x^A \frac{b_x^x}{b_x^z}$	$'t_{xh}^{z0} - \Delta_y^A \frac{b_x^y}{b_x^z}$	$'t_{xh}^{z1} - \Delta_y^A \frac{b_x^y}{b_x^z}$

Again, $\frac{3}{4}$ of all entering/leaving tests can be eliminated by noticing that $'t_{xh}^{\cdot\cdot}$ *entering* $\Leftrightarrow$ $''t_{xl}^{\cdot\cdot}$ *leaving* $\Leftrightarrow$ $t_{xh}^{\cdot\cdot}$ *entering*.

These new $'t$- and $''t$-values are consistent with the old t-values in the sense that $'t = 0$ and $''t = 0$ describe the same point.

The new origins of the left and right sub-boxes of B in B's own coordinate system are

$$
\begin{aligned}
q' &:= q \\
q'' &:= q + (c_x^B, 0, 0)
\end{aligned}
$$

In a similar manner, we have to split the s-tables of box B by using most of the values of the old s-values and computing the following new ones (see Figure 3).

$'s_{\cdot\cdot}^{\cdot\cdot} \rightarrow$, $\downarrow$	$y1$	$y3$	$z1$	$z3$
xl	$\frac{p_x - q''_x}{b_x^y}$	$'s_{xl}^{y1} - \Delta_z^B \frac{b_x^z}{b_x^y}$	$\frac{p_x - q''_x}{b_x^z}$	$'s_{xl}^{z1} - \Delta_y^B \frac{b_x^y}{b_x^z}$
xh	$'s_{xl}^{y1} + \frac{\Delta_x^A}{b_x^y}$	$'s_{xh}^{y1} - \Delta_z^B \frac{b_x^z}{b_x^y}$	$'s_{xl}^{z1} + \frac{\Delta_x^A}{b_x^z}$	$'s_{xh}^{z1} - \Delta_y^B \frac{b_x^y}{b_x^z}$
yl	$\frac{p_y - q''_y}{b_y^y}$	$'s_{yl}^{y1} - \Delta_z^B \frac{b_y^z}{b_y^y}$	$\frac{p_y - q''_y}{b_y^z}$	$'s_{yl}^{z1} - \Delta_y^B \frac{b_y^y}{b_y^z}$
yh	$'s_{yl}^{y1} + \frac{\Delta_y^A}{b_y^y}$	$'s_{yh}^{y1} - \Delta_z^B \frac{b_y^z}{b_y^y}$	$'s_{yl}^{z1} + \frac{\Delta_y^A}{b_y^z}$	$'s_{yh}^{z1} - \Delta_y^B \frac{b_y^y}{b_y^z}$
zl	$\frac{p_z - q''_z}{b_z^y}$	$'s_{zl}^{y1} - \Delta_z^B \frac{b_z^z}{b_z^y}$	$\frac{p_z - q''_z}{b_z^z}$	$'s_{zl}^{z1} - \Delta_y^B \frac{b_z^y}{b_z^z}$
zh	$'s_{zl}^{y1} + \frac{\Delta_z^A}{b_z^y}$	$'s_{zh}^{y1} - \Delta_z^B \frac{b_z^z}{b_z^y}$	$'s_{zl}^{z1} + \frac{\Delta_z^A}{b_z^z}$	$'s_{zh}^{z1} - \Delta_y^B \frac{b_z^y}{b_z^z}$

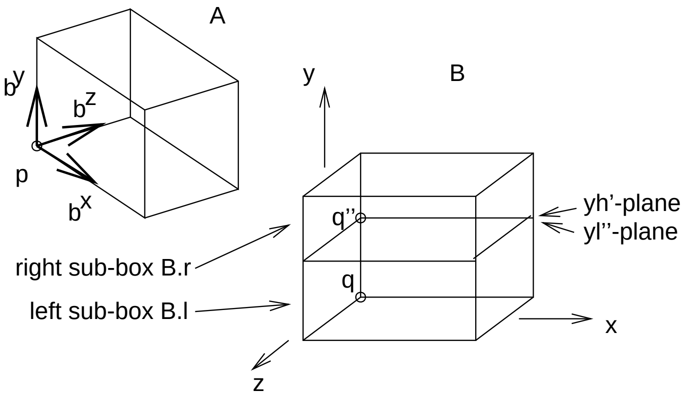

Figure 4. Splitting box B yields new line parameters for edges of A; cut-plane perpendicular to y-edges of B.

One can see, that only half of the terms $\Delta^B_{\cdot}\frac{b^{\cdot}_{\cdot}}{b^{\cdot}_{\cdot}}$ in the second and fourth column have to be computed.

The new $'S$- and $''S$-intervals are obtained from the old S-intervals by computing

$$\begin{array}{ll} 'S^{y1} = ''S^{y0} & 'S^{y3} = ''S^{y2} \\ 'S^{z1} = ''S^{z0} & 'S^{z3} = ''S^{z2} \end{array}$$

from scratch from the $'s$-values above; by shifting some S-intervals,

$$\begin{array}{lcl} ''S^{xi} & := & [\max\{0, S^{xi}_{\min} - c^B_x\}, S^{xi}_{\max} - c^B_x] \\ 'S^{xi} & := & [S^{xi}_{\min}, \min\{S^{xi}_{\max}, c^B_x\}] \end{array}$$

and by copying some:

$$\begin{array}{llll} 'S^{y0} := S^{y0} & 'S^{y2} := S^{y2} & 'S^{z0} := S^{z0} & 'S^{z2} := S^{z2} \\ ''S^{y1} := S^{y1} & ''S^{y3} := S^{y3} & ''S^{z1} := S^{z1} & ''S^{z3} := S^{z3} \end{array}$$

That is, 4 intervals (out of 12) have really to be computed.

The new origins of the left and right sub-boxes of B in A's local coordinate system are

$$\begin{array}{lcl} q' & := & q \\ q'' & := & q + c^B_x b^x \end{array}$$

Splitting B at c^B_y. This is quite similar to splitting it at c^B_x. We have to compute values $'t^{\alpha i}_{yh}$ (see Figure 4):

$\alpha\backslash i$	0	1	2	3
x	$\frac{q''_y - p_y}{b^x_y}$	$'t^{x0}_{yh} - \Delta^A_y \frac{b^y_y}{b^x_y}$	$'t^{x0}_{yh} - \Delta^A_z \frac{b^z_y}{b^x_y}$	$'t^{x1}_{yh} - \Delta^A_z \frac{b^z_y}{b^x_y}$
y	$\frac{q''_y - p_y}{b^y_y}$	$'t^{y0}_{yh} - \Delta^A_x \frac{b^x_y}{b^y_y}$	$'t^{y0}_{yh} - \Delta^A_z \frac{b^z_y}{b^y_y}$	$'t^{y1}_{yh} - \Delta^A_z \frac{b^z_y}{b^y_y}$
z	$\frac{q''_y - p_y}{b^z_y}$	$'t^{z0}_{yh} - \Delta^A_x \frac{b^x_y}{b^z_y}$	$'t^{z0}_{yh} - \Delta^A_y \frac{b^y_y}{b^z_y}$	$'t^{z1}_{yh} - \Delta^A_y \frac{b^y_y}{b^z_y}$

The new $'T$- and $''T$-intervals are obtained like above.

The new $'s$-values to be computed are (see Figure 5):

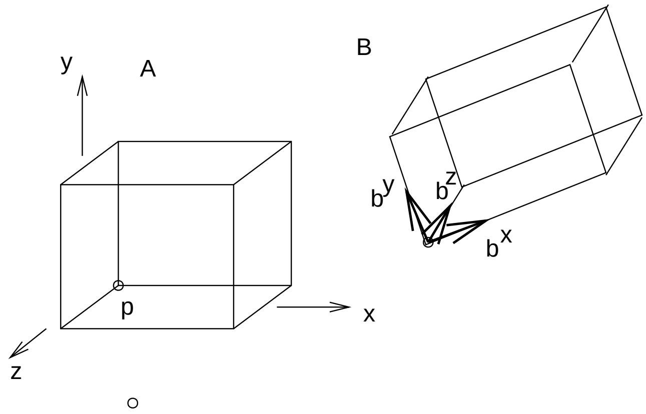

Figure 5. Splitting box B yields new line parameters for edges of B; cut-plane perpendicular to y-edges of B.

$'s^{\cdot\cdot}_{\cdot\cdot}$ $\to$ / $\downarrow$	$x1$	$x3$	$z2$	$z3$
xl	$\frac{p_x - q''_x}{b^x_x}$	$'s^{x1}_{xl} - \Delta^B_z \frac{b^z_x}{b^x_x}$	$\frac{p_x - q''_x}{b^z_x}$	$'s^{z2}_{xl} - \Delta^B_x \frac{b^x_x}{b^z_x}$
xh	$'s^{x1}_{xl} + \frac{\Delta^A_x}{b^x_x}$	$'s^{x1}_{xh} - \Delta^B_z \frac{b^z_x}{b^x_x}$	$'s^{z2}_{xl} + \frac{\Delta^A_x}{b^z_x}$	$'s^{z2}_{xh} - \Delta^B_x \frac{b^x_x}{b^z_x}$
yl	$\frac{p_y - q''_y}{b^x_y}$	$'s^{x1}_{yl} - \Delta^B_z \frac{b^z_y}{b^x_y}$	$\frac{p_y - q''_y}{b^z_y}$	$'s^{z2}_{yl} - \Delta^B_x \frac{b^x_y}{b^z_y}$
yh	$'s^{x1}_{yl} + \frac{\Delta^A_y}{b^x_y}$	$'s^{x1}_{yh} - \Delta^B_z \frac{b^z_y}{b^x_y}$	$'s^{z2}_{yl} + \frac{\Delta^A_y}{b^z_y}$	$'s^{z2}_{yh} - \Delta^B_x \frac{b^x_y}{b^z_y}$
zl	$\frac{p_z - q''_z}{b^x_z}$	$'s^{x1}_{zl} - \Delta^B_z \frac{b^z_z}{b^x_z}$	$\frac{p_z - q''_z}{b^z_z}$	$'s^{z2}_{zl} - \Delta^B_x \frac{b^x_z}{b^z_z}$
zh	$'s^{x1}_{zl} + \frac{\Delta^A_z}{b^x_z}$	$'s^{x1}_{zh} - \Delta^B_z \frac{b^z_z}{b^x_z}$	$'s^{z2}_{zl} + \frac{\Delta^A_z}{b^z_z}$	$'s^{z2}_{zh} - \Delta^B_x \frac{b^x_z}{b^z_z}$

From this table we compute

$$'S^{x1} = {}''S^{x0} \quad 'S^{x3} = {}''S^{x2}$$
$$'S^{z2} = {}''S^{z0} \quad 'S^{z3} = {}''S^{z1}$$

We copy

$$S^{x0}, S^{x2}, S^{z0}, S^{z1} \implies {}'S$$
$$S^{x1}, S^{x3}, S^{z1}, S^{z3} \implies {}''S$$

We shift/clip

$$''S^{yi} := \left[\max\{0, S^{yi}_{\min} - c^B_y\}, S^{yi}_{\max} - c^B_y\right]$$
$$'S^{yi} := \left[S^{yi}_{\min}, \min\{S^{yi}_{\max}, c^B_y\}\right]$$

The new origins of the left and right sub-boxes of B in B's own coordinate system are

$$q' := q$$
$$q'' := q + (0, c^B_y, 0)$$

and in A's local coordinate system they are

$$q'' := q + c^B_y b^y$$
$$q' := q$$

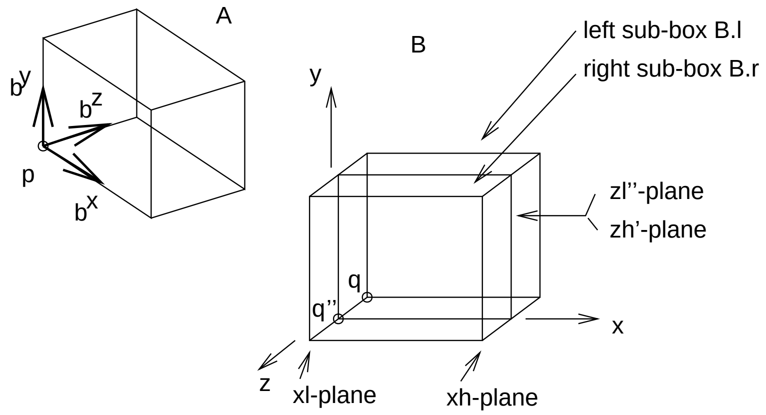

Figure 6. Splitting box B yields new line parameters for edges of A; cut-plane perpendicular to z-edges of B.

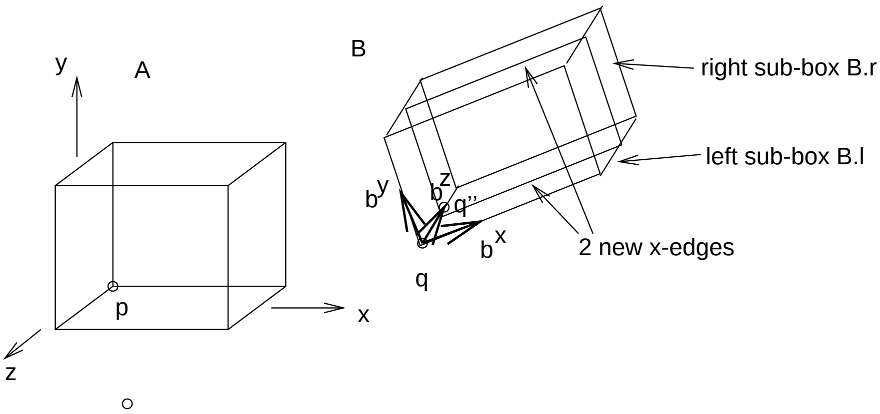

Figure 7. Splitting box B yields new line parameters for edges of B; cut-plane perpendicular to z-edges of B.

Splitting B at c_z^B. We have to compute values $'t_{zh}^{\alpha i}$ (see Figure 6):

$\alpha \backslash i$	0	1	2	3
x	$\frac{q''_z - p_z}{b_z^x}$	$'t_{zh}^{x0} - \Delta_y^A \frac{b_z^y}{b_z^x}$	$'t_{zh}^{x0} - \Delta_x^A \frac{b_z^z}{b_z^x}$	$'t_{zh}^{x1} - \Delta_x^A \frac{b_z^z}{b_z^x}$
y	$\frac{q''_z - p_z}{b_z^y}$	$'t_{zh}^{y0} - \Delta_z^A \frac{b_z^x}{b_z^y}$	$'t_{zh}^{y0} - \Delta_z^A \frac{b_z^z}{b_z^y}$	$'t_{zh}^{y1} - \Delta_y^A \frac{b_z^z}{b_z^y}$
z	$\frac{q''_z - p_z}{b_z^z}$	$'t_{zh}^{z0} - \Delta_z^A \frac{b_z^x}{b_z^z}$	$'t_{zh}^{z0} - \Delta_z^A \frac{b_z^y}{b_z^z}$	$'t_{zh}^{z1} - \Delta_y^A \frac{b_z^y}{b_z^z}$

See Figure 7:

$'s_{\downarrow}^{::\rightarrow}$	$x2$	$x3$	$y2$	$y3$
xl	$\frac{p_x - q''_x}{b_x^x}$	$'s_{xl}^{x2} - \Delta_y^B \frac{b_x^y}{b_x^x}$	$\frac{p_x - q''_x}{b_x^y}$	$'s_{xl}^{y2} - \Delta_x^B \frac{b_x^x}{b_x^y}$
xh	$'s_{xl}^{x2} + \frac{\Delta_x^A}{b_x^x}$	$'s_{xh}^{x2} - \Delta_y^B \frac{b_x^y}{b_x^x}$	$'s_{xl}^{y2} + \frac{\Delta_x^A}{b_x^y}$	$'s_{xh}^{y2} - \Delta_x^B \frac{b_x^x}{b_x^y}$
yl	$\frac{p_y - q''_y}{b_y^x}$	$'s_{yl}^{x2} - \Delta_y^B \frac{b_y^y}{b_y^x}$	$\frac{p_y - q''_y}{b_y^y}$	$'s_{yl}^{y2} - \Delta_x^B \frac{b_y^x}{b_y^y}$
yh	$'s_{yl}^{x2} + \frac{\Delta_y^A}{b_y^x}$	$'s_{yh}^{x2} - \Delta_y^B \frac{b_y^y}{b_y^x}$	$'s_{yl}^{y2} + \frac{\Delta_y^A}{b_y^y}$	$'s_{yh}^{y2} - \Delta_x^B \frac{b_y^x}{b_y^y}$
zl	$\frac{p_z - q''_z}{b_z^x}$	$'s_{zl}^{x2} - \Delta_y^B \frac{b_z^y}{b_z^x}$	$\frac{p_z - q''_z}{b_z^y}$	$'s_{zl}^{y2} - \Delta_x^B \frac{b_z^x}{b_z^y}$
zh	$'s_{zl}^{x2} + \frac{\Delta_z^A}{b_z^x}$	$'s_{zh}^{x2} - \Delta_y^B \frac{b_z^y}{b_z^x}$	$'s_{zl}^{y2} + \frac{\Delta_z^A}{b_z^y}$	$'s_{zh}^{y2} - \Delta_x^B \frac{b_z^x}{b_z^y}$

$'S^{x2} = ''S^{x0} \quad 'S^{x3} = ''S^{x1}$
$'S^{y2} = ''S^{y0} \quad 'S^{y3} = ''S^{y1}$

$$\begin{aligned} S^{x0}, S^{y0}, S^{x1}, S^{y1} &\implies 'S \\ S^{x2}, S^{x3}, S^{y2}, S^{y3} &\implies ''S \end{aligned}$$

$'S^{zi}$ and $''S^{zi}$ are shifted/clipped like above.